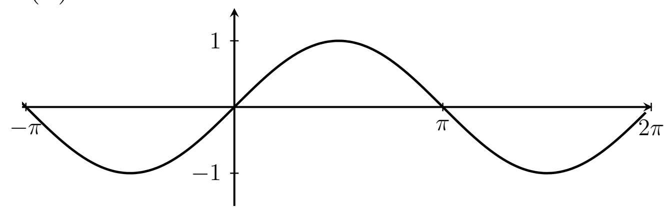
**Objectives:**

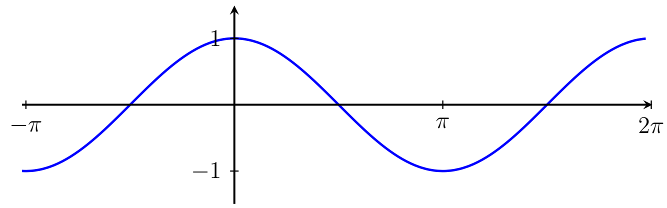
- Take derivatives of trigonometric functions.

**Intuition:** Let's take a look at $f(x) = \sin(x)$.

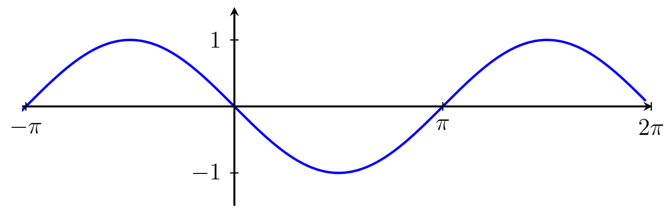
$f(x) = \sin(x)$




Guess: $f'(x) = \cos(x)$




Guess: $f''(x) = -\sin(x)$




**Derivative of $f(x) = \sin(x)$:**

$$\frac{d}{dx}\big(\sin(x)\big) = \cos(x)$$

*Proof:* Need $\lim_{h\to 0} \frac{\sin(h)}{h} = 1$ and $\lim_{h\to 0} \frac{\cos(h)-1}{h} = 0$ and the identity $\sin(a+b) = \sin a \cos b + \sin b \cos a$.

$$\begin{aligned} f'(x) = \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} &= \lim_{h\to 0} \frac{\sin(x)\cos(h)+\sin(h)\cos(x)-\sin(x)}{h} \\ &= \lim_{h\to 0} \frac{\sin(x)\cos(h)-\sin(x)}{h} + \lim_{h\to 0} \frac{\sin(h)\cos(x)}{h} \\ &= \lim_{h\to 0} \frac{\sin(x)\big(\cos(h)-1\big)}{h} + \lim_{h\to 0} \frac{\sin(h)}{h}\cos(x) \\ &= \sin(x)\lim_{h\to 0} \frac{\cos(h)-1}{h} + \cos(x)\lim_{h\to 0} \frac{\sin(h)}{h} \\ &= \sin(x)\cdot 0 + \cos(x)\cdot 1 \\ &= \cos(x). \end{aligned}$$

**Derivative of** $f(x) = \cos(x)$**:**

$$\frac{d}{dx}\big(\cos(x)\big) = -\sin(x)$$

*Proof:* You will need the identity $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$.

$$\begin{aligned} f'(x) = \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h} &= \lim_{h\to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h} \\ &= \lim_{h\to 0} \frac{\cos(x)\cos(h) - \cos(x)}{h} + \lim_{h\to 0} \frac{\sin(x)\sin(h)}{h} \\ &= \lim_{h\to 0} \frac{\cos(x)\big(\cos(h) - 1\big)}{h} + \lim_{h\to 0} \sin(x)\frac{\sin(h)}{h} \\ &= \cos(x) \lim_{h\to 0} \frac{\cos(h) - 1}{h} + \sin(x) \lim_{h\to 0} \frac{\sin(h)}{h} \\ &= \cos(x)\cdot 0 - \sin(x)\cdot 1 \\ &= -\sin(x). \end{aligned}$$

**More trig functions:** What are the derivatives of these trig functions?

- $f(x) = \tan(x) = \dfrac{\sin(x)}{\cos(x)}$ Use the quotient rule!

$$\begin{aligned} f'(x) &= \frac{\cos(x)\cos(x) - \sin(x)\big(-\sin(x)\big)}{\cos^2(x)} \\ &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\ &= \frac{1}{\cos^2(x)} \\ &= \sec^2(x) \end{aligned}$$

- $f(x) = \sec(x) = \dfrac{1}{\cos(x)}$

$$\begin{aligned} f'(x) &= \frac{\cos(x)\cdot 0 - 1\cdot\big(-\sin(x)\big)}{\cos^2(x)} \\ &= \frac{\sin(x)}{\cos^2(x)} \\ &= \frac{1}{\cos(x)}\cdot\frac{\sin(x)}{\cos(x)} \\ &= \sec(x)\tan(x) \end{aligned}$$

**Derivatives of trig functions:**

$\frac{d}{dx}\big(\sin(x)\big) = \cos(x)$ $\frac{d}{dx}\big(\cos(x)\big) = -\sin(x)$

$\frac{d}{dx}\big(\tan(x)\big) = \sec^2(x)$ $\frac{d}{dx}\big(\cot(x)\big) = -\csc^2(x)$

$\frac{d}{dx}\big(\sec(x)\big) = \sec(x)\tan(x)$ $\frac{d}{dx}\big(\csc(x)\big) = -\csc(x)\cot(x)$

**Examples:** Find the derivatives of the following functions.

1. $f(x) = x^2\sin(x)$

$$f'(x) = x^2\cos(x) + 2x\sin(x)$$

2. $f(x) = 2^x\tan(x)$

$$f'(x) = 2^x\sec^2(x) + \ln 2 \cdot 2^x\tan(x)$$

3. $f(x) = \dfrac{x+\sec(x)}{\sqrt{x}}$

$$f'(x) = \frac{\sqrt{x}\big(1+\sec(x)\tan(x)\big) - \big(x+\sec(x)\big)\frac{1}{2\sqrt{x}}}{x} = \frac{1}{\sqrt{x}}\big(1+\sec(x)\tan(x)\big) - \frac{1}{2x^{3/2}}\big(x+\sec(x)\big)$$

4. $f(x) = x^2e^x\cot x$

   This is the product of three functions. Use the product rule twice.

   $f'(x) = x^2\left(e^x(-\csc^2(x)) + \cot(x)e^x\right) + 2xe^x\cot(x) = -x^2e^x\csc^2(x) + x^2e^x\cot(x) + 2xe^x\cot(x)$.

   Note: $(fgh)' = fgh' + fg'h + f'gh$.

5. $f(x) = \cos\left(\sqrt{x}\right)$

   We can't take this derivative yet since this is the composition of functions.